On the partial Ricci curvature of foliations

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Abstract

We consider a problem of prescribing the partial Ricci curvature on a locally conformally flat manifold (M^n, g) endowed with the complementary orthogonal distributions D_1 and D_2. We provide conditions for symmetric $(0, 2)$-tensors T of a simple form (defined on M) to admit metrics $\tilde{g}$, conformal to g, that solve the partial Ricci equations. The solutions are given explicitly. Using above solutions, we also give examples to the problem of prescribing the mixed scalar curvature related to D_i. In aim to find "optimally placed" distributions, we calculate the variations of the total mixed scalar curvature (where again the partial Ricci curvature plays a key role), and give examples concerning minimization of a total energy and bending of a distribution.

Keywords and Phrases: Riemannian manifold, foliation, distribution, conformal, variation, Ricci curvature, scalar curvature, energy.

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Introduction

Let D_1 and D_2 be smooth complementary orthogonal distributions on a Riemannian manifold (M^n, g) with the Levi-Civita connection ∇ and the Riemannian curvature tensor R. Assume that $\dim D_i = p_i > 0$ $(i = 1, 2)$, $\dim M = n$, $n = p_1 + p_2$. Let $e_1, \dots e_n$ be a local orthonormal frame adapted to D_1 and D_2, i.e., $e_i \in D_1$ for $i \le p_1$ and $e_\alpha \in D_2$ for $\alpha > p_1$. The *mixed scalar curvature* is given by $K_{1,2} = \sum_{i,\alpha} g(R(e_i, e_\alpha)e_\alpha, e_i)$, see [8], [11]. We call the *partial Ricci curvature related to* D_1 the symmetric bilinear form $\mathrm{Ric}_1(x, y) = \sum_{\alpha > p_1} g(R(e_\alpha, x)y, e_\alpha)$ on the tangent bundle TM. The definition for $\mathrm{Ric}_2(x, y)$ is similar. Indeed,

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$K_{1,2} = \operatorname{Tr}_g \operatorname{Ric}_{i|\,D_i}$ for $i = 1, 2$. We are interested in the problem that concerns "optimally placed" distributions

($\boldsymbol{P}_0$) *Find variational formulae for the functional* $I_K : D_1 \to \int_M K_{1,2}\, d\,\mathrm{vol}$.

The variational formulae for I_K (and for i-th mean curvatures) related to codimension one distributions on a compact (M, g) are developed in [10]. In the paper we consider distribution D_1 of any codimension, represent the first and second variations of the total mixed scalar curvature and give examples concerning minimization of the total energy and bending of distributions. The partial Ricci curvatures play a key role in above variational formulae.

Different aspects of the problem of finding a metric g, whose Ricci tensor is a given second-order symmetric tensor T, were considered by several authors, see [1], [5] − [7], etc. The following problems for a differentiable manifold M with transversal complementary distributions D_1 and D_2, which generalize the classical ones, seem to be interesting:

($\boldsymbol{P}_1$) *Given a symmetric* $(0, 2)$*-tensor* T *on* M *satisfying* $T(D_1, D_2) = 0$*, does there exist a Riemannian metric* g *with the properties* $g(D_1, D_2) = 0$ *and either* $\operatorname{Ric}_{i\,|\,D_i} = T_{|\,D_i}$ *or* $[\operatorname{Ric}_i - \frac{1}{2}K_{1,2}\, g]_{\,|\,D_i} = T_{|\,D_i}$*, where* $i = 1, 2$ *?*

($\boldsymbol{P}_2$) *Given a function* $\bar K \in C(M)$*, does there exist a Riemannian metric* g *on* M*, whose mixed scalar curvature (related to* D_1 *and* D_2*) is* $\bar K$*?*

Note that ($\boldsymbol{P}_1$) for $T = 0$ asks about existence of either D_i-flat or "D_i-Einstein" metrics. ($\boldsymbol{P}_2$) is similar to the known problem of prescribing scalar curvature on M; its particular case $\bar K = const$ corresponds to the *Yamabe problem* (of prescribing constant scalar curvature on M).

We study the problems ($\boldsymbol{P}_1$) and ($\boldsymbol{P}_2$) on a locally conformally flat (M, g), in particular, on space forms, for tensors T of a simple form. We find necessary and sufficient conditions on T for the existence of metrics $\tilde g = (1/\phi^{\,2})\, g$ (conformal to the metric g) which solve the systems

$$a)\ \widetilde{\operatorname{Ric}}_{i\,|\,D_i} = T_{|\,D_i}\ (i = 1, 2); \quad b)\ [\,\widetilde{\operatorname{Ric}}_i - \frac{1}{2}\widetilde K_{1,2}\, \tilde g]_{\,|\,D_i} = T_{|\,D_i}\ (i = 1, 2). \tag{1}$$

The compatibility condition for (1)(a) is $\operatorname{Tr}_g T_{|\,D_1} = \operatorname{Tr}_g T_{|\,D_2}$ with the traces equal to $K_{1,2}$, while for (1)(b) is $(1 - p_2/2) \operatorname{Tr}_g T_{|\,D_1} = (1 - p_1/2) \operatorname{Tr}_g T_{|\,D_2}$.

In **Section 1** we determine all tensors T of ($\boldsymbol{P}_1$), the functions $\bar K$ of ($\boldsymbol{P}_2$) and the corresponding metrics $\tilde g$ that solve the systems (1). Theorems 1 − 4 and Corollaries 2 − 4 extend recent results of [5] − [7] (where $D_1 = TM$ and $D_2 = 0$) to cases of the partial Ricci and the mixed scalar curvatures of distributions. In **Section 2**, in aim to find "optimally placed" distributions, see ($\boldsymbol{P}_0$), we calculate the first and second variations of total $K_{1,2}$ using the partial Ricci curvature, and give examples concerning minimization of a total energy and bending of a distribution. **Section 3** contains proofs of results.

1 Prescribed partial Ricci curvature

We start with the solution to (1), see ($\boldsymbol{P}_1$), at a point $q \in M$. (The constant curvature metrics are solutions to $\mathrm{Ric}_{i\,|\,D_i} = \lambda_i\, g_{\,|\,D_i}$ at one point). Let M be a neighborhood of the origin O in $\mathbb{R}^n = \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, and D_i ($i = 1, 2$) is tangent to the i-th factor.

Proposition 1 *Let a diagonal n-by-n matrix T satisfies $\sum_{i\le p_1} T_{ii} = \sum_{\alpha>p_1} T_{\alpha\alpha}$. Then the metric g in a neighborhood of O in $\mathbb{R}^n = \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ satisfying (1)(a) at the origin, can be selected in the form $g = \sum_{a=1}^n \big(C - \sum_{b\le n} c_{bb} x_b^2\big) dx^a \oplus dx^a$, where, for example, $c_{ii} = T_{ii}/p_2$ ($1 \le i \le p_1$), $c_{\alpha\alpha} = (T_{\alpha\alpha} - \bar K/p_2)/p_1$ ($p_1 < \alpha \le n$), and $C > 0$ is large enough.*

Next we will represent the formulae relating partial Ricci and mixed scalar curvatures for two conformal each other metrics on M with distributions D_1, D_2. (Notice that the conformal change of a metric g preserves the orthogonality of D_1 and D_2.)

We call $\Delta^{(1)}\phi = \sum_{i\le p_1} h_\phi(e_i, e_i)$ and $\Delta^{(2)}\phi = \sum_{\alpha>p_1} h_\phi(e_\alpha, e_\alpha)$ the D_1- and D_2-*laplacian* of ϕ, resp., where h_ϕ is the Hessian of ϕ. (The Hessian of ϕ is the symmetric $(0,2)$-tensor $h_\phi(x, y) = g(S(x), y)$, where $S(x) = \nabla_x \nabla\phi$ is a self-adjoint $(1,1)$-tensor, and $\nabla\phi$ is the gradient of ϕ.) Indeed, $\Delta^{(1)} + \Delta^{(2)} = \Delta$. Let $\mathbb{R}^n = \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ ($p_1, p_2 > 0$) be decomposition of Euclidean space. The partial laplacians are $\Delta^{(1)} = \sum_{i\le p_1} \frac{\partial^2}{\partial x_i^2}$ and $\Delta^{(2)} = \sum_{\alpha>p_1} \frac{\partial^2}{\partial x_\alpha^2}$.

Proposition 2 *Let (M, g) be a Riemannian manifold with complementary orthogonal distributions D_1, D_2, and $\phi : M \to \mathbb{R}_+$ a smooth function. The partial Ricci curvatures and the mixed scalar curvature transform under conformal change of a metric $\tilde g = (1/\phi^2) g$ by the formulae*

$$\begin{aligned} \widetilde{\mathrm{Ric}}_1 &= \mathrm{Ric}_1 + \big[p_2 \phi\, h_\phi + \big(\phi\, \Delta^{(2)}\phi - p_2\, |\nabla\phi|^2\big) g\big]/\phi^{\,2}, \\ \widetilde{\mathrm{Ric}}_2 &= \mathrm{Ric}_2 + \big[p_1 \phi\, h_\phi + \big(\phi\, \Delta^{(1)}\phi - p_1\, |\nabla\phi|^2\big) g\big]/\phi^{\,2}, \qquad (2) \\ \widetilde K_{1,2} &= \phi^2 K_{1,2} + \phi\,(p_1\, \Delta^{(2)}\phi + p_2\, \Delta^{(1)}\phi) - p_1 p_2 |\nabla\phi|^2. \qquad (3) \end{aligned}$$

Corollary 1 *Given two complementary orthogonal distributions D_1 and D_2 on (M, g) with mixed scalar curvature K, let $\bar g = u^{2\gamma} g$ be a conformal metric, where $u > 0$ is a function on M, and $\gamma = (p_1 p_2/n - 1)^{-1}$. Then the mixed scalar curvature $\bar K$ of $\bar g$ satisfies the PDE*

$$-\,\gamma\big(p_1 \Delta^{(2)} + p_2 \Delta^{(1)}\big) u + K u = \bar K u^{2\gamma - 1}. \qquad (4)$$

Remark 1 One may extend above formulae for a pseudo-Riemannian metric. The formulae (2) and (4) are similar to the classical formulae (see, for example,

[6]) for the Ricci curvature

$$\widetilde{\mathrm{Ric}} - \mathrm{Ric} = \big[(n-2)\phi\, h_\phi + \big(\phi\,\Delta\phi - (n-1)\,|\nabla\phi|^2\big)g\big]/\phi^{\,2},$$
$$-4\,\frac{n-1}{n-2}\,\Delta u + Ku = \bar K u^{\frac{n+2}{n-2}}, \quad \bar g = u^{4/(n-2)} g.$$

We will consider the problem ($\boldsymbol{P}_1$) for a neighborhood $V \subset M^n$ of a locally conformally flat space. Suppose that there are coordinates $(x_1, \dots, x_n)$ on V with the metric $g_{ij} = \delta_{ij}/F^2$, where $F > 0$ is a differentiable function on M. We will fix on V canonical foliations $\mathcal{F}_1 = \{x_\alpha = c_\alpha,\ \alpha > p_1\}$ and $\mathcal{F}_2 = \{x_i = c_i,\ i \le p_1\}$, $c_i, c_\alpha \in \mathbb{R}$, consisting of coordinate submanifolds. Let $D_2 = T\mathcal{F}_1$ and $D_1 = T\mathcal{F}_2$ be their tangent distributions.

Theorem 1 *Let $(M^n, \bar g)$ be a locally conformally flat Riemannian manifold with complementary orthogonal distributions D_1, D_2 of dimensions $p_1, p_2 \ge 2$, and $V \subset M^n$ an open set with coordinates $(x_1, \dots, x_n)$ such that $\bar g_{ij} = \delta_{ij}/F^2$. Suppose that T is a symmetric $(0,2)$-tensor with the properties*

$$T_{ij} = f_1 \delta_{ij}, \quad T_{\alpha\beta} = f_2 \delta_{\alpha\beta}, \quad T_{i\alpha} = 0 \quad (i, j \le p_1,\ \alpha, \beta > p_1), \tag{5}$$

where $f_1, f_2 \in C^1(V)$. Then, in any of cases (a) or (b), there is a metric $\tilde g = (1/\phi^2)\bar g$ solving the problem (1) if and only if $\phi F = \sum_{i \le p_1}(a_1 x_i^2 + b_i x_i) + \sum_{\alpha > p_1}(a_2 x_\alpha^2 + b_\alpha x_\alpha) + c$, and

$$a)\ \ f_1 = -p_2[\lambda - 2(a_2 - a_1)\mu]/\phi^2, \quad f_2 = -p_1[\lambda - 2(a_2 - a_1)\mu]/\phi^2, \tag{6}$$
$$b)\ \ f_1 = \frac{p_2(p_1-2)}{2\,\phi^2}[\lambda - 2(a_2 - a_1)\mu], \quad f_2 = \frac{p_2(p_1-2)}{2\,\phi^2}[\lambda - 2(a_2 - a_1)\mu]. \tag{7}$$

Here $a_1, a_2, b_k, c \in \mathbb{R}$, and

$$\lambda = (\sum\nolimits_k b_k^2) - 2(a_1 + a_2)\,c, \ \ \mu(x) = \sum\nolimits_i (a_1 x_i^2 + b_i x_i) - \sum\nolimits_\alpha (a_2 x_\alpha^2 + b_\alpha x_\alpha). \tag{8}$$

Corollary 2 *Let D_1, D_2 be tangent distributions to canonical foliations on the Euclidean product space $(\mathbb{R}^n = \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}, g)$, and $p_1, p_2 \ge 2$. Suppose that T is a symmetric $(0,2)$-tensor satisfying (5), where $f_1, f_2 \in C^1(\mathbb{R}^n)$. Then there is a metric $\tilde g = (1/\phi^2)g$ solving the problem (1) (in any of cases (a) or (b)) if and only if $\phi = \sum_{i \le p_1}(a_1 x_i^2 + b_i x_i) + \sum_{\alpha > p_1}(a_2 x_\alpha^2 + b_\alpha x_\alpha) + c$, and f_a, f_β are given by (6), (7), where $a_1, a_2, b_k, c \in \mathbb{R}$, and λ, $\mu(x)$ are defined in (8). A non-complete metric $\tilde g$ is defined on $\mathbb{R}^n$, if either $a_1, a_2 > 0$ and $\frac{1}{a_1}\sum_i b_i^2 + \frac{1}{a_2}\sum_\alpha b_\alpha^2 < 4c$ or $a_1, a_2 < 0$ and $\frac{1}{a_1}\sum_i b_i^2 + \frac{1}{a_2}\sum_\alpha b_\alpha^2 > 4c$. In other cases, excluding the homothety, $\tilde g$ has singular points.*

Example 1 If $a_i = a$ in Corollary 2, then $\widetilde K_{1,2} = -p_1 p_2 \lambda$ defined in (8), and the singularity set of $\tilde g$ can be explicitly described in terms of λ. Namely, if $\lambda < 0$

then a non-complete metric $\tilde{g}$ is defined on $\mathbb{R}^n$, and if $\lambda \geq 0$ then, excluding the homothety, the set of singularity points of $\tilde{g}$ consists of 1) a point if $\lambda = 0$; 2) a hyperplane if $\lambda > 0$ and $a = 0$; 3) an $(n-1)$-dimensional sphere if $\lambda > 0$ and $a \neq 0$.

One may consider the pseudo-Euclidean space $(\mathbb{R}^n, g)$ with the coordinates $x = (x_1, \ldots, x_n)$ and the metric $g_{km} = \epsilon_k \delta_{km}$, $\epsilon_k = \pm 1$. Then, for example, in case (a) of Corollary 2 for $a_1 = a_2 = a$, we have $\phi = \sum_{k=1}^n (\epsilon_k a x_k^2 + b_k x_k) + c$, where $p_1 \sum_{\alpha > p_1} \epsilon_\alpha = p_2 \sum_{i \leq p_1} \epsilon_i$, and $f_1 = -p_2 \lambda/\phi^2 - (2a/\phi)(p_2 - \sum_\alpha \epsilon_\alpha)$, $f_2 = -p_1 \lambda/\phi^2 - (2a/\phi)(p_1 - \sum_i \epsilon_i)$.

Example 2 We will discuss our results, when M is the hyperbolic space $(\mathbb{H}^n, \bar{g})$, represented by the half space model $\mathbb{R}^n_+$, $x_n > 0$, and $\bar{g}_{ij} = \delta_{ij}/x_n^2$. For any pair of integers $p_1, p_2 > 0$, $p_1 + p_2 = n$, denote by $\mathcal{F}$ a foliation by p_2-planes $\{x\} \times \mathbb{R}^{p_2}$, where $x = (x_1, \ldots, x_{p_1}, 0, \ldots, 0)$. Let D_2 be the distribution tangent to $\mathcal{F}$, and D_1 its orthogonal complement. Using $F = x_n$ in Theorem 1, we obtain $\phi\, x_n = \sum_{i \leq p_1} (a_1 x_i^2 + b_i x_i) + \sum_{\alpha > p_1} (a_2 x_\alpha^2 + b_\alpha x_\alpha) + c$.

If $a_1 = a_2 = a$, the mixed scalar curvature $\widetilde{K}_{1,2} = -p_1 p_2 \lambda$. Moreover, in this case a non-complete metric $\tilde{g}$ is defined on $\mathbb{H}^n$ whenever (i) $\lambda < 0$; (ii) $\lambda = 0$ and $a = 0$ (hence $c \neq 0$); (iii) $\lambda = 0$, $a \neq 0$ and $b_n/a \geq 0$; (iv) $\lambda > 0$, $a = 0$, $b_k = 0$ for $k < n$ and $c/b_n \geq 0$; (v) $\lambda > 0$, $a \neq 0$ and $b_n/a \geq \sqrt{\lambda}/|a|$. Otherwise, the singularity set of $\tilde{g}$ consists of intersection of a hyperplane or a sphere with the half-space $x_n > 0$.

We show (i)–(v) in case a). From (3) and $\mathrm{Ric}_i = 0$, we get $\widetilde{K}_{1,2} = -p_1 p_2 \lambda$.

If $\lambda < 0$ then $\tilde{g}$ is defined on $\mathbb{H}^n$ and T is positive definite. Let $\lambda = 0$. If $a = 0$ then $b_k = 0$ for $k < n$, $\phi = c/x_n \neq 0$ and $\tilde{g}$ is defined on $\mathbb{H}^n$. If $a \neq 0$, then if $b_n/(2a) \geq 0$ then $\tilde{g}$ is defined on $\mathbb{H}^n$; otherwise, if $b_n/(2a) < 0$ then $\tilde{g}$ has a singularity at the point $\tilde{x} = -(b_1, \ldots b_n)/(2a)$.

Let $\lambda > 0$. If $a = 0$, we have two cases. In the first one, we have $b_k = 0$ for $k < n$ and $b_n \neq 0$. In this case, if $c/b_n \geq 0$ then $\tilde{g}$ is defined on $\mathbb{H}^n$; otherwise if $c/b_n < 0$, then any point of the hyperplane $x_n = -c/b_n$ is a singularity point of $\tilde{g}$. In the second case, we have $b_{k_0} \neq 0$ for some $k_0 < n$, then any point that belongs to the intersection of the hyperplane $(\sum_k b_k x_k) + c = 0$ with the half-space $x_n > 0$, is a singularity point of $\tilde{g}$.

When $\lambda > 0$ and $a \neq 0$, if $b_n/a \geq \sqrt{\lambda}/|a|$, then $\tilde{g}$ is defined on $\mathbb{H}^n$. Otherwise, if $b_n/a < \sqrt{\lambda}/|a|$, then any point p of the $(n-1)$-dimensional sphere centered at the point of coordinates $\tilde{x} = -(b_1, \ldots b_n)/(2a)$, with radius $\sqrt{\lambda}/(2|a|)$, such that p is the half space $x_n > 0$, is a point of singularity of $\tilde{g}$. The case b) is similar to case a).

The next theorem extends Theorem 1 to the tensors T such that $T_{|D_1} = \sum_{i \leq p_1} f_i(x_k)\, dx_i^2$ and $T_{|D_2} = \sum_{\alpha > p_1} f_\alpha(x_k)\, dx_\alpha^2$, with a fixed index $k \leq p_1$.

Theorem 2 *Let $(M^n,\bar g)$ be a locally conformally flat Riemannian manifold with complementary orthogonal distributions D_1,D_2 of dimensions $p_1,p_2\ge 3$. Let $V\subset M^n$ be an open set with coordinates $(x_1,\dots,x_n)$ such that $\bar g_{ij}=\delta_{ij}/F^2$. Consider a symmetric $(0,2)$-tensor T satisfying*

$$T_{ij}=f_i(x_k)\delta_{ij},\quad T_{\alpha\beta}=f_\alpha(x_k)\delta_{\alpha\beta},\quad T_{i\alpha}=0\quad (i,j\le p_1,\ \alpha,\beta>p_1),\tag{9}$$

with a fixed $k\le p_1$. Suppose that the functions $f_i,f_\alpha\in C^1(V)$ are not all constants and are not all equal. Then, in any of cases (a) or (b), there is a metric $\tilde g=\bar g/\phi^2$ solving the problem (1) if and only if there is a differentiable function $U(x_k)$ on V such that $\phi F=e^U$ and

$$\begin{aligned}
&a)\ f_k=p_2U'',\quad f_i=-p_2U'^2\ (i\ne k),\quad f_\alpha=U''-(p_1{-}1)U'^2\ (\alpha\ge p_1),\\
&b)\ f_k=\frac12 p_2\big(U''+(p_1{-}1)U'^2\big),\quad f_i=-\frac12 p_2\big(U''-(p_1{-}3)U'^2\big)\ (i\ne k),\\
&\qquad f_\alpha=\frac12(p_2-2)[(p_1-1)U'^2-U'']\ (\alpha\ge p_1).
\end{aligned}\tag{10}$$

Corollary 3 *Let D_1,D_2 be tangent distributions to canonical foliations on the Euclidean product space $(\mathbb{R}^n=\mathbb{R}^{p_1}\times\mathbb{R}^{p_2},\,g)$ with $p_1,p_2\ge 3$. Consider a symmetric $(0,2)$-tensor T with the properties (9) for a fixed $k\le p_1$. Suppose that the functions $f_i,f_\alpha\in C^1(\mathbb{R}^n)$ are not all constants and are not all equal. Then, in any of cases (a) or (b), there is a metric $\tilde g=(1/\phi^2)g$ solving the problem (1) if and only if there is a differentiable function $U(x_k)$ such that $\phi=e^U$ and (10) holds. If $\phi\le C$ for some constant $C>0$, then the metrics are complete on $\mathbb{R}^n$.*

Example 3 (i) In the case (a) of Theorem 2, assuming that all functions f_i,f_α are constant, we obtain $U=ax_k+b$ and $\phi=e^{ax_k+b}$, where $a,b\in\mathbb{R}$.

(ii) Consider the function $U=-x_k^{2m}$ for some fixed $k\le p_1$ and $m\in\mathbb{N}$. In conditions of Corollary 3, case (a), we obtain $f_k=-2m(2m-1)p_2x_k^{2m-2}\le 0$, $f_i=-4m^2p_2x_k^{4m-2}\le 0$ and $f_\alpha=-2mx_k^{2m-2}[2m-1+2m(p_1-1)x_k^{2m}]\le 0$. Hence $\widetilde{\mathrm{Ric}}_i\le 0$ $(i=1,2)$. By Corollary 3, the metric $\tilde g$ is complete on $\mathbb{R}^n$.

(iii) Consider the periodic function $U=\sin x_k$ for some fixed $k\le p_1$. In both cases of Corollary 3, the metric $\tilde g$ is periodic in all variables, and it can be considered as a complete metric on a cylinder or an n-dimensional torus. The mixed scalar curvature of $\tilde g$, $\widetilde K_{1,2}=-p_2\,e^{2\sin x_k}[\sin x_k+(p_1-1)\cos x_k]$, takes positive and negative values. Case (b), $\phi=e^{\sin x_k}$, can be considered as an example of tensors T defined on a flat torus with a pair of complementary distributions D_i, that admits a solution to the case (b) of (1).

(iv) From Theorem 2, with $F=x_n$, we obtain results for a half-space $(\mathbb{R}^n_+,\bar g)$ with the hyperbolic metric $\bar g_{ij}=\delta_{ij}/x_n^2$. If $U=-x_n^{2m}$, where $m\in\mathbb{N}$, then $\bar g=\bar g/\phi^2$ is a complete metric on $\mathbb{R}^n_+$ and the partial Ricci curvatures are negative, the calculations are similar to (ii).

In Theorem 3, $T_{|D_1} = \sum_{i\le p_1} f_i(x_k, x_\delta)\,dx_i^2$ and $T_{|D_2} = \sum_{\alpha>p_1} f_\alpha(x_k, x_\delta)\,dx_\alpha^2$ have fixed indices $k \le p_1$ and $\delta > p_1$.

Theorem 3 *Let $(M^n, \bar g)$ be a locally conformally flat Riemannian manifold with complementary orthogonal distributions D_1, D_2 of dimensions $p_1, p_2 \ge 3$. Let $V \subset M^n$ be an open set with coordinates $(x_1, \ldots, x_n)$ such that $\bar g_{ij} = \delta_{ij}/F^2$. Given $k \le p_1$ and $\delta > p_1$, consider a symmetric $(0,2)$-tensor T with the properties $T_{ij} = f_i(x_k, x_\delta)\delta_{ij}$, $T_{\alpha\beta} = f_\alpha(x_k, x_\delta)\delta_{\alpha\beta}$, and $T_{i\alpha} = 0$ $(i, j \le p_1,\ \alpha, \beta > p_1)$, where the functions $f_i, f_\alpha \in C^1(V)$ are not all constants and are not all equal. Then, in any of cases (a) or (b), there is a metric $\tilde g = \bar g/\phi^2$ solving the problem (1) if and only if there are differentiable functions $v(x_k), w(x_\delta)$ such that $\phi F = v + w$, where*

$$\begin{aligned} a)\ \ & f_k = [(p_2 v'' + w'')(v+w) - p_2\,(v'^2 + w'^2)]/(v+w)^2,\\ & f_\delta = [(v'' + p_1 w'')(v+w) - p_1\,(v'^2 + w'^2)]/(v+w)^2,\\ & f_\alpha = f_\delta - p_1 w''/(v+w)\ (\forall\,\alpha \ne \delta),\ \ f_i = f_k - p_2 v''/(v+w)\ (\forall\, i \ne k), \end{aligned} \tag{11}$$

$$\begin{aligned} b)\ \ & f_k = [\,\tfrac12((2-p_2)v'' + p_1 w'')(v+w) + p_2(p_1-1)\,(v'^2 + w'^2)]/(v+w)^2,\\ & f_\delta = [\,\tfrac12(p_2 v'' - (p_1-2)w'')(v+w) + p_1(p_2-1)\,(v'^2 + w'^2)]/(v+w)^2,\\ & f_\alpha = f_\delta - p_1 w''/(v+w)\ (\forall\,\alpha \ne \delta),\ \ f_i = f_k - p_2 v''/(v+w)\ (\forall\, i \ne k). \end{aligned} \tag{12}$$

Theorem 4 *Let $(M^n, \bar g)$ be a locally conformally flat Riemannian manifold with complementary orthogonal distributions D_1, D_2 of dimensions $p_1, p_2 \ge 3$. Let $V \subset M^n$ be an open set with coordinates $(x_1, \ldots, x_n)$ such that $\bar g_{ij} = \delta_{ij}/F^2$. Consider a non-diagonal symmetric $(0,2)$-tensor T with the properties $T_{ij} = f_{ij}(x_i, x_j)$, $T_{\alpha\beta} = f_\alpha(x_\alpha, x_\beta)$, and $T_{i\alpha} = 0$ $(i, j \le p_1,\ \alpha, \beta > p_1)$, where $f_{AB} \in C^1(V)$. Suppose that the functions $f_i, f_\alpha \in C^1(V)$ are not all constants and are not all equal. Then, there is a metric $\tilde g = \bar g/\phi^2$ solving the problem (1)(a) if and only if up to a change of order of D_1 and D_2, one of the following cases occur:*

(i) *$f_{12}(x_1, x_2)$ is any nonzero differentiable function, $f_{ij} \equiv 0$ for all $i \ne j$ such that $i \ge 3$ or $j \ge 3$ and $\phi\,F = \varphi(x_1, x_2)$ is a non-vanishing solution to the PDE $\varphi_{,x_1x_2} = (f_{12}/p_2)\varphi$.*

(ii) *There is an integer $p \in [3, p_1]$ such that $f_{ij} = 0$, if $i \ne j$, $i \ge p+1$ or $j \ge p+1$. Moreover, there exist non-constant differentiable functions, $U_j(x_j)$, for $1 \le j \le p$ such that for all i, j, $1 \le i \ne j \le p$ one of the following holds:*

$$f_{ij} = p_2 U_i' U_j' \quad \text{and} \quad \phi\,F = a\,e^{\sum_{j=1}^p U_j} + b\,e^{-\sum_{j=1}^p U_j}, \tag{13}$$

$$f_{ij} = -p_2 U_i' U_j' \quad \text{and} \quad \phi\,F = a\cos\Big(\sum\nolimits_{j=1}^p U_j\Big) + b\sin\Big(\sum\nolimits_{j=1}^p U_j\Big), \tag{14}$$

where $a, b \in \mathbb{R}$ and $a^2 + b^2 > 0$. Moreover, in each case ϕ is defined on an open connected subset of V, where it does not vanish. (The solution to (1)(b) is constructed similarly.)

Remark 2 If $(M^n, \bar{g})$ is the Euclidean space, and $|v(x_k)|$, $|w(x_\delta)| \le C$ and $0 < |F\phi(x)| \le C$ for some constant $C > 0$, then the metrics given in Theorems 3 and 4 are complete on $\mathbb{R}^n$.

By considering $u = (\phi F)^{\,1-p_1p_2/n}$ and the mixed scalar curvature $\tilde{K}$ obtained from the partial Ricci tensor $T_{|\,D_i})$, as a consequence of Theorems 1−4, case (a), we present C^∞ solutions to the non-linear PDE's of the type (4)

$$(p_1\Delta^{(2)} + p_2\Delta^{(1)})u + (1 - p_1p_2/n)\tilde{K}u^{-\frac{1+p_1p_2/n}{1-p_1p_2/n}} = 0, \quad p_1 + p_2 = n. \tag{15}$$

We will show that for certain functions $\tilde{K}$, depending on functions of one variable, or an arbitrary constant, there exist conformally flat metrics $\tilde{g}$, whose mixed scalar curvature is $\tilde{K}$, see $(\boldsymbol{P}_2)$.

Corollary 4 *Let $\widetilde{K}$ is given by*

(i) $-p_1p_2[\lambda + 2(a_2 - a_1)\,\mu(x)]$, *where $a_1, a_2, b_k, c \in \mathbb{R}$, and λ, $\mu(x)$ are defined in (8).*

(ii) $p_2\, e^{2U}[U'' - (p_1{-}1)U'^2]$, *where $U(x_k)$ is a differentiable function, for some $k \le p_1$.*

(iii) $(v{+}w)(p_2v''{+}p_1w''){-}p_1p_2(v'^2{+}w'^2)$, *where $v(x_k)$, $w(x_\delta)$ are differentiable functions, for some $k \le p_1$, $\delta \ge p_1$.*

(iv) $p_2(af{-}bf^{-1})\big[\sum_j(U_j'^2{+}U_j''){+}p_1\frac{af-bf^{-1}}{af+bf^{-1}}\sum_j U_j'^2\big]$, *where $U_j(x_j)$ $(1 \le j \le p)$ are non-constant differentiable functions, $3 \le p \le p_1$, $a^2{+}b^2 > 0$ and $f = e^{\sum_j U_j}$.*

Then (15) has a solution, globally defined on $\mathbb{R}^n$, given, resp., by

(i) $u = \big[\sum_{i\le p_1}(a_1x_i^2 + b_ix_i) + \sum_{\alpha>p_1}(a_2x_\alpha^2 + b_\alpha x_\alpha) + c\big]^{1-p_1p_2/n}$.

(ii) $u = e^{(1-p_1p_2/n)U}$.

(iii) $u = (v + w)^{1-p_1p_2/n}$.

(iv) $u = (af + bf^{-1})^{1-p_1p_2/n}$. *(If $a = 1$ and $b = 0$, then $\widetilde{K} = p_2e^{\sum_j U_j}\big[(p_1 + 1)\sum_j U_j'^2 + \sum_j U_j''\big]$ and (15) has a solution $u = e^{\,(1-p_1p_2/n)\sum_j U_j}$. A solution to (15) corresponding to (14) is constructed similarly).*

2 The variational formulae for the total mixed scalar curvature

Now let $D_1 = D$ and $D_2 = D^\perp$ be a pair of complementary orthogonal distributions on a compact Riemannian manifold (M, g).

Definition 1 Let $p_1 \le p_2$, and δ_{ij} the Kronecker symbol. For any point $q \in M$ and orthonormal bases e_i $(i \le p_1)$ of $D_1(q)$ and ξ_j $(j \le p_2)$ of $D_1(q)^\perp$, consider the bilinear form $\mathcal{I}_q(\vec{\omega}, \vec{\omega}) = \sum_{i,j=1}^{p_1} \Phi_{ij}\, \omega_i\, \omega_j$ with the coefficients

$$\begin{aligned}\Phi_{ij} = {} & [(\mathrm{Ric}_1 - \mathrm{Ric}_2)(\xi_j, \xi_j) - (\mathrm{Ric}_1 - \mathrm{Ric}_2)(e_i, e_i)]\,\delta_{ij}\\ & +2[\,g(R(e_i, e_j)\xi_i, \xi_j) + g(R(e_i, \xi_j)\xi_i, e_j)].\end{aligned} \tag{16}$$

We say that $\mathcal{I}$ is *quasi-positive* if $\mathcal{I}_q$ is positive definite for arbitrary adapted orthonormal basis $\{e_i, \xi_j\}$ at any $q \in M \setminus \Sigma$, where Σ is a set of zero volume.

In next theorem, using the partial Ricci curvature, we calculate the first and second variations of the *total mixed scalar curvature* $I_K : D \to \int_M K_{D,D^\perp}\, d\,\mathrm{vol}$ of a distribution of arbitrary dimension $p,\ 1 \le p < \dim M$.

Theorem 5 *A distribution D on a compact Riemannian manifold (M, g) is a critical point for the functional I_K if and only if*

$$(\mathrm{Ric}_1 - \mathrm{Ric}_2)(D, D^\perp) = 0. \tag{17}$$

It is a point of local minimum if the form $\mathcal{I}$ (of Definition 1) is quasi-positive.

Let $\mathrm{codim}\, D = 1$, and $N \perp D$ be a unit vector field on a domain $V \subset M$. Then $\mathrm{Ric}_2(D, D^\perp) = 0$ and (17) is reduced to $\mathrm{Ric}(x, N) = 0$ $(x \in D)$. Only one term of $\mathcal{I}$ is presented: $\Phi_{11} = \mathrm{Ric}(\xi_1, \xi_1) - \mathrm{Ric}(N, N)$. Hence, $\mathcal{I}$ is positive definite at $q \in M$ if and only if $\mathrm{Ric}(x, x) - \mathrm{Ric}(N, N)|x|^2 > 0$ for all non-zero $x \perp N$ in T_qM. With this remark, we have the following.

Corollary 5 ([10]) *A unit vector field N orthogonal to a codimension-one distribution D on a compact Riemannian manifold (M, g) is a critical point for the functional $I_2 : N \to \int_M \mathrm{Ric}(N, N)\, d\,\mathrm{vol}$, if and only if*

$$\mathrm{Ric}(N, x) = 0 \quad \forall\, x \in D. \tag{18}$$

It is a point of local minimum if the form $\mathcal{I}_{2,N}$ is positive definite on the space of sections of D. The above bilinear form on the space of vector fields orthogonal to N is given by $\mathcal{I}_{2,N}(x, y) = \mathrm{Ric}(x, y) - \mathrm{Ric}(N, N)\, g(x, y)$.

Example 4 (a) If D and $D^\perp$ are curvature invariant, then $\mathrm{Ric}_i(D, D^\perp) = 0$ $(i = 1, 2)$, hence D is critical for I_K. (A distribution D is called *curvature invariant* if $R(x, y)z \in D$ for all $x, y, z \in D$.)

(b) For Einstein manifold M^4 of non-constant sectional curvature, any "optimally placed" two-dimensional distribution consists of planes with maximal or minimal curvature. To see this, one may use the following characteristic property of Einstein 4-manifolds among Riemannian manifolds (M^4, g): "the sectional curvature $K(Q) = K(Q^\perp)$ for any 2-plane Q", see Corollary 1.129 in [1]. (Hence, the product $S^2(1) \times \mathbb{H}^2(-1)$ is not Einstein manifold, namely, $\mathrm{Ric}(e_1, e_1) = 1$, $\mathrm{Ric}(e_3, e_3) = -1$ when $e_1 \in TS^2$, $e_3 \in T\mathbb{H}^2$.)

The *co-nullity tensor* $C : D^\perp \times D \to D$ of a distribution D assigns to a pair (ξ, x), x being tangent and ξ normal to D, the tangent (to D) component of the vector field $\nabla_x \xi$.

The $2k$*-th mean curvature* of a distribution D^p is the integral

$$\sigma_{2k}(D(q)) = \frac{p}{\mathrm{vol}(S^{p-1})}\int_{\xi\perp D(q),\,|\xi|=1}\sigma_{2k}(C(\xi,\cdot))\,d\,\omega,\quad \text{for all}\ \ q\in M.$$

The same formula determines $\sigma_{2k-1}(D(q))=0$. Denote by $C^{\alpha}_{1,ij}{=}g(C^{\alpha}_1 e_i, e_j)$ and $C^{i}_{2,\alpha\beta}{=}g(C^{i}_2 e_\alpha, e_\beta)$, where the linear operators $C^\alpha_1:\mathbb{R}^{p_1}\to\mathbb{R}^{p_1}$ and $C^i_2:\mathbb{R}^{p_2}\to\mathbb{R}^{p_2}$ correspond to co-nullity tensors C_i of D_i at $q\in M$.

Proposition 3 *Let* D_i, $\dim D_i = p_i$ $(i=1,2)$ *be a pair of complementary orthogonal distributions on a compact Riemannian manifold* (M,g)*. Then*

$$I_K = 2\int_M \sigma_2(D_1)+\sigma_2(D_2)\,d\,\mathrm{vol}\,.$$

The extremal values of I_K can be used for estimation of the total energy and bending of a distribution or a vector field.

We can regard the distribution D_i as the map $\tilde D_i: M\to G(p_i,M)$ (section of the Grassmann bundle $G(p_i,M)=\cup_{q\in M}G(p_i,T_qM)$), where $\tilde D_1(q)=e_1\wedge\cdots\wedge e_{p_1}$ and $\tilde D_2(q)=e_{p_1+1}\wedge\cdots\wedge e_m$ are the p_i-vectors determined locally by $D_1(q)$ and $D_2(q)$, resp. For a map between Riemannian spaces $f:\bar M\to(M,g)$, the *energy* is defined to be $\mathcal{E}(f)=\frac12\int_M\sum_{a=1}^m g(d\,f(e_a),d\,f(e_a))\,d\,\mathrm{vol}$ (see [3]). The *corrected energy* of a p_2-dimensional distribution D_2 on a $(p_1{+}p_2)$-dimensional Riemannian manifold (M,g) is defined in [2] as

$$\mathcal{D}(D_2)=\int_M\sum\nolimits_{a=1}^m|\nabla_{e_a}\tilde D_2|^2+p_1(p_1-2)\,|H_1|^2+p_2^2\,|H_2|^2\,d\,\mathrm{vol},$$

where $|d\tilde D_2|$ is calculated from the definition of Sasaki metric g_s:

$$\sum\nolimits_{a=1}^m g_s(d\tilde D_2(e_a),d\tilde D_2(e_a))=\sum\nolimits_{a=1}^m[\,g(e_a,e_a)+g(\nabla_{e_a}\tilde D_2,\nabla_{e_a}\tilde D_2)]$$

and $H_1=-\frac{1}{p_1}\sum_\alpha(\sum_i C^\alpha_{1,ii})\,e_\alpha$, $H_2=-\frac{1}{p_2}\sum_i(\sum_\alpha C^i_{2,\alpha\alpha})\,e_i$. If D_2 is integrable, then $\mathcal{D}(D_2)\ge\int_M K_{1,2}\,d\,\mathrm{vol}$, [2]. Similarly, we define the *total bending* $\mathcal{B}(D_2)=c_n\int_M|\nabla\tilde D_2|^2\,d\,\mathrm{vol}$, where c_n is a constant.

Proposition 4 *The total bending of a* p_1*-dimensional distribution* D_1 *on a* (p_1+p_2)*-dimensional Riemannian manifold* M *satisfies the inequality*

$$\mathcal{B}(D_1)\ge c_n\int_M\frac{2}{p_1{-}1}\sum\nolimits_\alpha\sigma_2(C^\alpha_1)+\frac{2}{p_2{-}1}\sum\nolimits_i\sigma_2(C^i_2)\,d\,\mathrm{vol}\qquad(19)$$

that for $p_1=p_2=p$ *takes the form* $\mathcal{B}(D_1)\ge\frac{c_n}{p-1}\,I_K$.

3 Proof of results

Proof of Proposition 1. The partial Ricci curvature in local coordinates is

$$\begin{aligned}\mathrm{Ric}_1(g)_{ij} &= \tfrac12\sum\nolimits_{\alpha\beta} g^{\alpha\beta}(g_{i\alpha,j\beta}+g_{j\alpha,i\beta}-g_{ij,\alpha\beta}-g_{\alpha\beta,ij})+Q_1(g,\partial g),\\ \mathrm{Ric}_2(g)_{\alpha\beta} &= \tfrac12\sum\nolimits_{ij} g^{ij}(g_{\alpha i,\beta j}+g_{\beta i,\alpha j}-g_{\alpha\beta,ij}-g_{ij,\alpha\beta})+Q_2(g,\partial g),\end{aligned}$$

where g^{ab} is the inverse of g_{ab} and Q_i is a function of g and its derivatives, and is homogeneous of degree 2 in the first derivatives of g. Since $g_{i\alpha}\equiv 0$, we obtain

$$\begin{aligned}\mathrm{Ric}_1(g)_{ij} &= -\tfrac12\sum\nolimits_{\alpha\beta} g^{\alpha\beta}(g_{ij,\alpha\beta}+g_{\alpha\beta,ij})+Q_1(g,\partial g),\\ \mathrm{Ric}_2(g)_{\alpha\beta} &= -\tfrac12\sum\nolimits_{ij} g^{ij}(g_{\alpha\beta,ij}+g_{ij,\alpha\beta})+Q_2(g,\partial g).\end{aligned}\tag{20}$$

Assume that g has the form $g=\sum_{a=1}^n\big(C+\sum_{b\le n}c_{bb}x_b^2\big)dx^a\oplus dx^a$. Then $g_{ii,\alpha\alpha}=2\,c_{\alpha\alpha}$, $g_{\alpha\alpha,ii}=2\,c_{ii}$. Substituting in (20) and using (1), we get at O

$$p_2c_{ii}+\sum\nolimits_\alpha c_{\alpha\alpha}=T_{ii}\ (1\le i\le p_1),\quad p_1c_{\alpha\alpha}+\sum\nolimits_i c_{ii}=T_{\alpha\alpha}\ (p_1<\alpha\le n).\tag{21}$$

The summing of first p_1 equations and last p_2 ones in (21) yields $p_1\sum_\alpha c_{\alpha\alpha}+p_2\sum_i c_{ii}=K$, where $K=\sum_{i\le p_1}T_{ii}=\sum_{\alpha>p_1}T_{\alpha\alpha}$. The linear system (21) consists of n equations and the same number of variables, its rank $<n$. It is easily seen that, for instance, $c_{ii}=T_{ii}/p_2$ $(1\le i\le p_1)$ and $c_{\alpha\alpha}=(T_{\alpha\alpha}-K/p_2)/p_1$ $(p_1<\alpha\le n)$ is a solution (satisfying $\sum_\alpha c_{\alpha\alpha}=0$). The tensor g is positive definite at O when $C>0$ is large enough. $\square$

Proof of Proposition 2. Define a metric $\tilde g$ on M by $\tilde g=e^\psi g$, where $\psi=-2\log\phi$. The connections ∇ and $\tilde\nabla$ of g and $\tilde g$ are related by [4] $\tilde\nabla_x y=\nabla_x y+\frac12(x(\psi)\,y+y(\psi)\,x-g(x,y)\nabla\psi)$. Let $\tilde R$ be the curvature tensor of $\tilde g$. Using known formulae (see, for example, [4]) we obtain

$$\begin{aligned}\tilde R(x,y)y &= R(x,y)y+\frac12(g(h_\psi(x),y)y-g(h_\psi(y),y)x-|y|^2h_\psi(x))\\ &+\frac14([\,y(\psi)^2-|y|^2|\nabla\psi|^2\,]x+[\,x(\psi)y(\psi)-y(\psi)|\nabla\psi|^2\,]y+x(\psi)\,|y|^2\nabla\psi),\end{aligned}\tag{22}$$

where $x\in D_2, y\in D_1$, hence $g(x,y)=0$. If z is a g-unit vector then $\tilde z=z\,e^{-\psi/2}$ is a $\tilde g$-unit vector. From this and (22), using adapted orthonormal base $\{e_i,e_\alpha\}$, one may deduce the relation between partial Ricci curvatures in both metrics

$$\begin{aligned}\widetilde{\mathrm{Ric}}_1(e_A,e_B) &= \mathrm{Ric}_1(e_A,e_B)-\frac12(\delta_{AB}\sum\nolimits_\alpha g(h_\psi(e_\alpha),e_\alpha)+p_2g(h_\psi(e_A),e_B))\\ &+\frac14([\sum\nolimits_\alpha g(\nabla\psi,e_\alpha)^2-p_2|\nabla\psi|^2]\delta_{AB}+p_2g(\nabla\psi,e_A)\,g(\nabla\psi,e_B)).\end{aligned}$$

In matrix notation, this reads as

$$\widetilde{\mathrm{Ric}}_1=\mathrm{Ric}_1-\frac12\big((\Delta^{(2)}\psi)\,g+p_2h_\psi\big)+\frac14\big(\big[|\nabla^{(2)}\psi|^2-p_2|\nabla\psi|^2\big]g+p_2g(\nabla.\psi,\nabla.\psi)\big)\tag{23}$$

where $\nabla^{(2)}$ and $\Delta^{(2)}$ are D_2-gradient and D_2-laplacian of a function.

To short the formulae, we will turn back to the function $e^\psi = 1/\phi^2$. In this case $\nabla\psi = -\frac{2}{\phi}\nabla\phi$, $h_\psi(e_A, e_B) = \frac{2}{\phi^2}\, g(\nabla_{e_A}\phi, \nabla_{e_B}\phi) - \frac{2}{\phi}\, h_\phi(e_A, e_B)$, $\Delta^{(2)}\psi = \frac{2}{\phi^2}\, |\nabla^{(2)}\phi|^2 - \frac{2}{\phi}\, \Delta^{(2)}\phi$, etc. Substituting above equalities in (23), we obtain

$$\begin{aligned}\widetilde{\mathrm{Ric}}_1 - \mathrm{Ric}_1 \,=\, & -\big(\frac{1}{\phi^{\,2}}|\nabla^{(2)}\phi|^2 - \frac{1}{\phi}\,\Delta^{(2)}\phi\big)g - p_2\big(\frac{1}{\phi^{\,2}}g(\nabla_{\cdot}\phi, \nabla_{\cdot}\phi) - \frac{1}{\phi}\,h_\phi\big)\\ & + \frac{1}{\phi^{\,2}}|\nabla^{(2)}\phi|^2 g - \frac{p_2}{\phi^{\,2}}|\nabla\phi|^2 g + \frac{p_2}{\phi^{\,2}}g(\nabla_{\cdot}\phi, \nabla_{\cdot}\phi)\end{aligned}$$

that is simplified to (2). The formula for D_2 is proved similarly. By $\widetilde{K}_{1,2} = \phi^2\sum_{i\le p_1}\widetilde{\mathrm{Ric}}_1(e_i, e_i)$, (3) is the result of the trace operation applied to (2). $\square$

Proof of Corollary 1. Define the function $u = \phi^{1-p_1p_2/n}$. Then $\phi = u^{-\gamma}$ and $\Delta^{(i)}\phi = \gamma u^{\gamma-1}\Delta^{(i)}u + \gamma(\gamma-1)\, u^{\gamma-2}|\nabla^{(i)}u|^2$ $(i = 1, 2)$, $|\nabla\phi|^2 = \gamma^2 u^{2\gamma-2}|\nabla u|^2$. Substituting in (3), we obtain (4). $\square$

Proof of Theorem 1. a) The compatibility condition for (P_1) is $p_1f_1 = p_2f_2$. Observe that $\tilde g = \bar g/\phi^2 = g/(\phi F)^2 = g/\varphi^2$, where g is the Euclidean metric, and $\varphi = \phi F$. In view of $\mathrm{Ric}_1 = \mathrm{Ric}_2 = 0$ for g, we have, see (2),

$$\begin{aligned}\widetilde{\mathrm{Ric}}_1 &= \big[p_2\varphi\, h_\varphi + \big(\varphi\,\Delta^{(2)}\varphi - p_2\,|\nabla\varphi\,|^2\big)g\big]/\varphi^{\,2},\\ \widetilde{\mathrm{Ric}}_2 &= \big[p_1\varphi\, h_\varphi + \big(\varphi\,\Delta^{(1)}\varphi - p_1\,|\nabla\varphi\,|^2\big)g\big]/\varphi^{\,2}.\end{aligned} \tag{24}$$

Since $T_{|D_k} = \widetilde{\mathrm{Ric}}_{k\,|D_k}$ $(k = 1, 2)$, we obtain

$$\begin{aligned}\varphi^{\,2}f_1 g &= p_2\varphi\, h_\varphi + \big(\varphi\,\Delta^{(2)}\varphi - p_2\,|\nabla\varphi|^2\big)g \quad \text{on } D_1,\\ \varphi^{\,2}f_2 g &= p_1\varphi\, h_\varphi + \big(\varphi\,\Delta^{(1)}\varphi - p_1\,|\nabla\varphi|^2\big)g \quad \text{on } D_2.\end{aligned}$$

Hence, the problem is reduced to studying the following system of PDE's:

$$\begin{aligned}p_2\varphi_{,x_ix_i} &= \varphi f_1 - \Delta^{(2)}\varphi + p_2\,|\nabla\varphi\,|^2/\varphi, \quad i \le p_1\\ p_1\varphi_{,x_\alpha x_\alpha} &= \varphi f_2 - \Delta^{(1)}\varphi + p_1\,|\nabla\varphi\,|^2/\varphi, \quad \alpha > p_1,\\ \varphi_{,x_kx_m} &= 0, \quad 1 \le k \ne m \le n.\end{aligned} \tag{25}$$

From the last equation of (25) we conclude that $\varphi = \sum_{k=1}^n \phi_k(x_k)$. From the first two equations of (25) we deduce $\phi_i''(x_i) = 2a_1 \in \mathbb{R}$ for all $i \le p_1$ and $\phi_\alpha''(x_\alpha) = 2a_2 \in \mathbb{R}$ for all $\alpha > p_1$. Therefore, $\varphi = \sum_{i\le p_1}(a_1x_i^2 + b_ix_i) + \sum_{\alpha>p_1}(a_2x_\alpha^2 + b_\alpha x_\alpha) + c$, where $b_k, c \in \mathbb{R}$. We also have $\Delta^{(1)}\varphi = 2a_1p_1$ and $\Delta^{(2)}\varphi = 2a_2p_2$. Hence the first two equations of (25) are reduced to

$$2p_2(a_1 + a_2) = \varphi f_1 + p_2\,|\nabla\varphi|^2/\varphi, \quad 2p_1(a_1 + a_2) = \varphi f_2 + p_1\,|\nabla\varphi|^2/\varphi.$$

Comparing them, we see that the equality $p_1f_1 = p_2f_2\, (= \phi^2\widetilde{K}_{1,2})$ is necessary for the solution existence. In view of $|\nabla\varphi|^2 - 2(a_1 + a_2)\varphi = \lambda - 2(a_2 - a_1)\,\mu$, we obtain f_1 and f_2, as required.

If $a_1 a_2 \le 0$ and $a_1^2 + a_2^2 + \sum_k b_k^2 > 0$ then the set $\{\phi = 0\}$ of singularities of $\tilde g$ is non-empty (and can be explicitly described). If $a_1 a_2 > 0$ then the inequality $\phi > 0$ means that the discriminant of a quadratic equation is negative.

b) The proof is similar to the previous one. The compatibility condition for (P_2) is $(1 - p_2/2)p_1 f_1 = (1 - p_1/2)p_2 f_2$. The problem is reduced to the following system of PDE's:

$$
\begin{aligned}
p_2\varphi_{,x_i x_i} &= \varphi f_1 - \Delta^{(2)}\varphi + p_2\,|\nabla\varphi|^2/\varphi + \widetilde K_{1,2}/(2\varphi), \quad i \le p_1\\
p_1\varphi_{,x_\alpha x_\alpha} &= \varphi f_2 - \Delta^{(1)}\varphi + p_1\,|\nabla\varphi|^2/\varphi + \widetilde K_{1,2}/(2\varphi), \quad \alpha > p_1,\\
\varphi_{,x_k x_m} &= 0, \quad 1 \le k \ne m \le n,
\end{aligned}
\tag{26}
$$

where $\varphi = F\phi$. From the last equation of (26) we conclude that $\varphi = \sum_{k=1}^n \phi_k(x_k)$. Moreover, from the first two equations of (26) we deduce $\varphi_i''(x_i) = 2a_1 \in \mathbb{R}$ for all $i \le p_1$ and $\varphi_\alpha''(x_\alpha) = 2a_2 \in \mathbb{R}$ for all $\alpha > p_1$. Therefore, $\varphi = \sum_{i \le p_1}(a_1 x_i^2 + b_i x_i) + \sum_{\alpha > p_1}(a_2 x_\alpha^2 + b_\alpha x_\alpha) + c$, where $b_k, c \in \mathbb{R}$. We also calculate $\Delta^{(1)}\varphi = 2a_1 p_1$ and $\Delta^{(2)}\varphi = 2a_2 p_2$. Hence the first two equations of (26) are reduced to the equations

$$
\begin{aligned}
2p_2(a_1 + a_2)\varphi &= \varphi^2 f_1 + p_2\,|\nabla\varphi|^2 + \widetilde K_{1,2}/2,\\
2p_1(a_1 + a_2)\varphi &= \varphi^2 f_2 + p_1\,|\nabla\varphi|^2 + \widetilde K_{1,2}/2.
\end{aligned}
$$

In view of $|\nabla\varphi|^2 - 2(a_1 + a_2)\varphi = \lambda - 2(a_2 - a_1)\,\mu(x)$, we obtain f_i as required.$\square$

Proof of Corollary 2. a) We set $F = 1$ and obtain ϕ, f_1, f_2 as in the proof of Theorem 1. Assume that $a_1 = a_2 = a$ (see Example 1). Then we have $\phi = \sum_{i=k}^n (a x_k^2 + b_k x_k) + c$, and $f_1 = -p_2\lambda/\phi^2$, $f_2 = -p_1\lambda/\phi^2$, $\widetilde K_{1,2} = -p_1 p_2 \lambda$. If $\lambda < 0$ then $a \ne 0$ and $\phi > 0$. Namely, $\phi \ge c - \frac{1}{4a}\sum_k b_k^2 > 0$, if $a > 0$, and $\phi \le c + \frac{1}{4a}\sum_k b_k^2 < 0$, if $a < 0$. Let $\lambda = 0$. If $a \ne 0$ then $\phi = 0$ has a unique solution at $\tilde x = -(b_1, \dots b_n)/(2a)$, hence $\tilde g$ has one singular point. If $a = 0$ then $b_k = 0$. Hence $\phi = c$ and $\tilde g$ reduces to a homothety.

Now let $\lambda > 0$. If $a = 0$ then there is k_0 such that $b_{k_0} \ne 0$ and ϕ vanishes on the hyperplane $(\sum_k b_k x_k) + c = 0$ – a singularity set of $\tilde g$. If $a \ne 0$ then ϕ vanishes on the $(n-1)$-dimensional sphere, centered at $\tilde x = -(b_1, \dots b_n)/(2a)$ with radius $\sqrt{\lambda}/(2|a|)$ – a singularity set of $\tilde g$.

b) The proof is similar to the previous one. $\square$

Proof of Theorem 2. a) We set $\tilde g = \bar g/\phi^2 = g/(\phi F)^2 = g/\varphi^2$, where g is the Euclidean metric, and $\varphi = \phi F$. From (1)(a), (2), in view of $\mathrm{Ric}_1 = \mathrm{Ric}_2 = 0$ for g, we have (24). Hence, the problem reduces to studying the following system of differential equations:

$$
\begin{aligned}
p_2\varphi_{,x_i x_i} &= \varphi f_i - \Delta^{(2)}\varphi + p_2\,|\nabla\varphi|^2/\varphi, \quad i \le p_1,\\
p_1\varphi_{,x_\alpha x_\alpha} &= \varphi\, f_\alpha - \Delta^{(1)}\varphi + p_1\,|\nabla\varphi|^2/\varphi, \quad \alpha > p_1,\\
\varphi_{,x_k x_m} &= 0, \quad 1 \le k \ne m \le n,
\end{aligned}
\tag{27}
$$

From the third equation of (27) we conclude that $\varphi = \sum_{s=1}^{n} \phi_s(x_s)$, which substituted in the first two equations gives

$$\begin{aligned} p_2\phi_i''(x_i) &= \varphi f_i - \Delta^{(2)}\varphi + p_2\,|\nabla\varphi|^2/\varphi, \quad i \le p_1, \\ p_1\phi_\alpha''(x_\alpha) &= \varphi\, f_\alpha - \Delta^{(1)}\varphi + p_1\,|\nabla\varphi|^2/\varphi, \quad \alpha > p_1, \end{aligned} \tag{28}$$

where $f_i = f_i(x_k)$, $f_\alpha = f_\alpha(x_k)$. As a consequence of (28) we have

$$\begin{aligned} p_2(\phi_k''(x_k) - \phi_i''(x_i)) &= \varphi(f_k - f_i), \quad i \le p_1, \\ p_1(\phi_\alpha''(x_\alpha) - \phi_\beta''(x_\beta)) &= \varphi(f_\alpha - f_\beta), \quad \alpha, \beta > p_1, \end{aligned} \tag{29}$$

and then

$$\frac{\phi_k''(x_k) - \phi_i''(x_i)}{\phi_k''(x_k) - \phi_j''(x_j)} = \frac{f_k - f_i}{f_k - f_j}\ (i, j \le p_1), \quad \frac{\phi_\alpha''(x_\alpha) - \phi_\beta''(x_\beta)}{\phi_\alpha''(x_\alpha) - \phi_\gamma''(x_\gamma)} = \frac{f_\alpha - f_\beta}{f_\alpha - f_\gamma}\ (\alpha, \beta, \gamma > p_1). \tag{30}$$

From $(30)_1$ (not all f_i are equal), in view of $p_1, p_2 \ge 3$, we deduce that φ is the function of x_k only, i.e., $\varphi = \phi_k(x_k)$. Next, from $(29)_2$ we obtain $f_\alpha = f_\beta$ $(\forall\, \alpha, \beta)$ and $f_i = f_k - p_2\phi_k''/\phi_k$ $(\forall\, i \ne k)$. Then we calculate $|\nabla\varphi|^2 = {\phi_k'}^2$, $\Delta^{(1)}\varphi = \phi_k''$ and $\Delta^{(2)}\varphi = 0$. Hence, $f_k = p_2\frac{\phi_k''\phi_k - {\phi_k'}^2}{(\phi_k)^2} = p_2(\log\phi_k)''$. Assuming $f_k = p_2 U''(x_k)$ we immediately obtain $\phi_k = e^{U(x_k)}$ and $\phi = \varphi/F = \frac{1}{F}e^{U(x_k)}$ that is required. Next, we confirm that $f_i = -p_2 {U'}^2$ $(i \ne k)$ and $f_\alpha = U'' - (p_1 - 1){U'}^2$. One may verify that the compatibility condition for T, $f_k + (p_1 - 1)f_i = p_2 f_\alpha$ holds, is satisfied. Finally, $\widetilde{K}_{1,2} = \varphi^2 p_2 f_\alpha = p_2 e^{2U}[U'' - (p_1{-}1){U'}^2]$.

b) The proof is similar to the previous one. The problem is reduced to studying the system

$$\begin{aligned} p_2\varphi_{,x_i x_i} &= \phi f_i - \Delta^{(2)}\varphi + p_2\,|\nabla\varphi|^2/\varphi + \widetilde{K}_{1,2}/(2\varphi), \quad i \le p_1, \\ p_1\phi_{,x_\alpha x_\alpha} &= \varphi f_\alpha - \Delta^{(1)}\varphi + p_1\,|\nabla\varphi|^2/\varphi + \widetilde{K}_{1,2}/(2\varphi), \quad \alpha > p_1, \\ \phi_{,x_k x_m} &= 0, \quad 1 \le k \ne m \le n. \end{aligned} \tag{31}$$

As in the case a) we obtain $\phi = \sum_{s=1}^{n} \phi_s(x_s)$, and

$$\begin{aligned} p_2\phi_i''(x_i) &= \varphi f_i(x_k) - \Delta^{(2)}\varphi + p_2\,|\nabla\varphi|^2/\varphi + \widetilde{K}_{1,2}/(2\varphi), \quad i \le p_1, \\ p_1\phi_\alpha''(x_\alpha) &= \varphi f_\alpha(x_k) - \Delta^{(1)}\varphi + p_1\,|\nabla\varphi|^2/\varphi + \widetilde{K}_{1,2}/(2\varphi), \quad \alpha > p_1. \end{aligned} \tag{32}$$

As a consequence of (32) we again have (29), (30). Similarly to case a), we conclude that $\varphi = \phi_k(x_k)$. Next, from (29) we obtain $f_\alpha = f_\beta$ $(\forall\, \alpha, \beta)$ and $f_i = f_k - p_2\phi_k''/\phi_k$ $(\forall\, i \ne k)$. Then we calculate $|\nabla\varphi|^2 = {\phi_k'}^2$, $\Delta^{(1)}\varphi = \phi_k''$ and $\Delta^{(2)}\varphi = 0$. Hence, $\widetilde{K}_{1,2}(x) = p_2[\phi_k''\phi_k - p_1{\phi_k'}^2]$. From (32) with $i = k$ we obtain

$$f_k = \frac{p_2}{\phi_k^{\,2}}\big[(\phi_k''\phi_k - {\phi_k'}^2) - (\phi_k''\phi_k - p_1{\phi_k'}^2)/2\big] = \frac{p_2}{2}[(\log\phi_k)'' + (p_1 - 1)(\phi_k'/\phi_k)^2].$$

Assuming $\phi_k = e^{U(x_k)}$ we immediately obtain $f_k = \frac{1}{2}p_2(U'' + (p_1-1)U'^2)$ that is required. Next, we obtain $f_i = -\frac{1}{2}p_2(U'' - (p_1-3)U'^2)$ $(i \ne k)$ and $f_\alpha = \frac{1}{2}(p_2-2)[-U'' + (p_1-1)U'^2]$. One may verify that the compatibility condition for T, $(1-p_2/2)[f_k + (p_1-1)f_i] = (1-p_1/2)p_2 f_\alpha$, is satisfied. $\square$

Proof of Corollary 3. We consider $F = 1$ and apply the arguments similar to those of Theorem 2. The metric $\tilde g$, satisfying $\phi(x_k) \le C$, is complete, since there exists a constant $m > 0$, such that $|v|_{\tilde g} \ge m|v|$ for any $v \in \mathbb{R}^n$. $\square$

Proof of Theorem 3. a) We set $\varphi = F\phi$, and as in the proof of Theorem 2, obtain $\varphi = \sum_{s=1}^n \phi_s(x_s)$. Similarly to the proof of Theorem 2, we deduce (28) – (30), where $f_i = f_i(x_k, x_\delta)$, $f_\alpha = f_\alpha(x_k, x_\delta)$. From these (not all f_i, f_α are equal), in view of $p_1, p_2 \ge 3$, we have that φ is the function of x_k, x_δ only, i.e., $\varphi = u + v$, where $u = \phi_k(x_k)$ and $v = \phi_\delta(x_\delta)$. Hence

$$f_\alpha = f_\delta - p_1 w''/(u+v) \;\; (\forall\, \alpha \ne \delta), \quad f_i = f_k - p_2 u''/(u+v) \;\; (\forall\, i \ne k). \tag{33}$$

Then we find $|\nabla\varphi|^2 = u'^2 + v'^2$, $\Delta^{(1)}\varphi = u''$ and $\Delta^{(2)}\varphi = w''$. Hence, the functions u, v satisfy the system (11). The compatibility condition for T,

$$p_2(p_1-1)u'' - p_1(p_2-1)w'' = (p_1 f_k - p_2 f_\delta)(u+v) \tag{34}$$

that is the linear combination of equations in (11) with coefficients p_1 and p_2.

b) As in proof of the case a), we conclude that $\varphi = \sum_{s=1}^n \phi_s(x_s)$. Similarly to the proof of case a), we deduce (28) – (30), As a consequence of these we have, where $f_i = f_i(x_k, x_\delta)$, $f_\alpha = f_\alpha(x_k, x_\delta)$. As in a), we deduce that φ is the function of x_k, x_δ only, i.e., $\varphi = u + v$, where $u = \phi_k(x_k)$, $v = \phi_\delta(x_\delta)$. Next, we obtain (33). Then we calculate $|\nabla\varphi|^2 = u'^2 + v'^2$, $\Delta^{(1)}\varphi = u''$ and $\Delta^{(2)}\varphi = w''$. The mixed scalar curvature is $\tilde K_{1,2} = (u+v)(p_2 u'' + p_1 w'') - p_1 p_2(u'^2 + v'^2)$. Hence, the functions u, v satisfy the nonlinear system (12). The compatibility condition for T takes the form $p_2(p_1-1)(p_1-2)u'' - p_1p_2(p_2-2)w'' = (p_1(p_1-2)f_k - p_2(p_2-2)f_\delta)(u+v)$ that is the linear combination of equations in (12). $\square$

Lemma A [7] *Assume $\varphi(x_1, \dots, x_p)$, $p \ge 3$, is a non-vanishing differentiable function that satisfies a system of equations*

$$\varphi_{,ij} = f_{ij}\,\varphi, \quad i \ne j, \tag{35}$$

where $f_{ij} = f_{ji}$ is a differentiable function of x_i and x_j. Assume there is an open subset $V \subset \mathbb{R}^p$, where all f_{ij} do not vanish. Then there is an open dense subset of V where $\prod_i \varphi_{,i}$ does not vanish. On each connected component of this subset, there exist differentiable functions $U_i(x_i) \ne 0$ $(i = 1, \dots, p)$ such that $f_{ij} = U_i(x_i)\,U_j(x_j)$, $1 \le i \ne j \le p$.

Lemma B [7] *A non-vanishing differentiable function $\varphi(x_1, \dots, x_p)$, $p \ge 3$,*

is a solution to $$\begin{aligned} &\varphi_{,ij} - \varphi = 0 \;(i \ne j) \;\Leftrightarrow\; \varphi = a\, e^{\sum_{j=1}^p x_j} + b\, e^{-\sum_{j=1}^p x_j},\\ &\varphi_{,ij} + \varphi = 0 \;(i \ne j) \;\Leftrightarrow\; \varphi = a\cos(\textstyle\sum_{j=1}^p x_j) - b\sin(\textstyle\sum_{j=1}^p x_j),\end{aligned}$$

where $a, b \in \mathbb{R}$ and $a^2 + b^2 > 0$.

Proof of Theorem 4. We set $\tilde{g} = \bar{g}/\phi^2 = g/(\phi F)^2 = g/\varphi^2$, where g is the Euclidean metric, and $\varphi = \phi F$. From (1)(a), (2), in view of $\mathrm{Ric}_i = 0$ for g, we have (24). By conditions, we get the system of PDE's

$$\begin{aligned} p_2\,\varphi_{,x_ix_i} &= \varphi f_{ii} - \Delta^{(2)}\varphi + p_2\,|\nabla\varphi|^2/\varphi \quad (i \le p_1), \\ p_1\varphi_{,x_\alpha x_\alpha} &= \varphi\, f_{\alpha\alpha} - \Delta^{(1)}\varphi + p_1|\nabla\varphi|^2/\varphi \quad (\alpha > p_1), \\ p_2\,\varphi_{,x_ix_j} &= f_{ij}\,\varphi \quad (1 \le i \ne j \le p_1), \\ p_1\varphi_{,x_\alpha x_\beta} &= f_{\alpha\beta}\,\varphi \quad (p_1 < \alpha \ne \beta \le n), \\ \varphi_{,x_ix_\alpha} &= 0 \quad (1 \le i \le p_1,\ p_1 < \alpha \le n). \end{aligned} \tag{36}$$

Since T is non-diagonal, there is a pair i_0, j_0 (assume them $\le p_1$) such that $f_{i_0j_0} \ne 0$ on an open set $V_1 \subset V$. From $(36)_{3,4}$, since f_{ij} are functions of two variables, and $p_1, p_2 \ge 3$, it follows

$$f_{ij}\varphi_{,k} = f_{ik}\varphi_{,j} = f_{jk}\varphi_{,i}\ (= \varphi_{,ijk}/p_2), \quad \text{for all } i, j, k \text{ distinct.} \tag{37}$$

<u>If $f_{i_0k} \equiv 0$ on V_1, for all k distinct from i_0 and j_0</u>, then we may assume (under a change of indices) that $f_{12}(x_1, x_2) \ne 0$ and $f_{1k} \equiv 0$ for $3 \le< k \le n$ on V_1. We have $f_{12}\varphi_{,\alpha} = \varphi_{,12\alpha} = (\varphi_{,1\alpha})_{,2} = 0$, moreover, by (37), $f_{12}\varphi_{,i} = f_{1i}\varphi_{,2} = f_{2i}\varphi_{,1}$. Hence, $\varphi_{,k} = f_{2k} = 0$ for all $2 < k \le n$.

Observe that $\varphi_{,1}$ and $\varphi_{,2}$ cannot be zero on any open subset of V_1, otherwise we would have $\varphi_{,12} = f_{12}\varphi/p_2 = 0$. This is a contradiction since φ is a non-vanishing function. Therefore, there exists an open subset $V_2 \subset V_1$, where $\varphi_{,1} \ne 0$ and $\varphi_{,2} \ne 0$. Hence, $f_{2k} \equiv 0$ on V_2 for all $k \ge 3$. From (37) we get $f_{2j}\varphi_{,k} = f_{jk}\varphi_{,2}$, for $3 \le j \ne k \le p_1$ and therefore $f_{jk} \equiv 0$ on V_2. Differentiating $(36)_3$, yields $f_{\alpha\beta}\varphi_{,1} = p_1\varphi_{,\alpha\beta 1} = 0$, and using $\varphi_{,1} \ne 0$, we conclude that $f_{\alpha\beta} \equiv 0$ for all $p_1 < \alpha \ne \beta \le n$ on V_2.

Finally, we have that φ depends on x_1 and x_2 only, and is a solution to the PDE $\varphi_{,x_1x_2} = (f_{12}/p_2)\varphi$ in the (x_1, x_2)-plane. Moreover, $(36)_{1,2}$ determine the diagonal elements of T which will depend on (x_1, x_2) only.

<u>Otherwise</u>, there exist distinct indices i, j, k (assume them $\le p_1$) such that f_{ij} and f_{ik} do not vanish on an open subset V_1 of V. Observe that $\varphi_{,k}$ and $\varphi_{,j}$ cannot be zero on any open subset of U, since φ is a non-vanishing differentiable function, see $(36)_3$. Let $V_2 \subset V_1$ be an open subset where $\varphi_{,k} \ne 0$ and $\varphi_{,j} \ne 0$. It follows from (37), $f_{jk} \ne 0$ and $\varphi_{,i} \ne 0$ on V_2. By reordering the variables, if necessary, we may consider $i = 1$ and $f_{1j} \ne 0$, on an open subset $V_3 \subset V_2$, for all j, such that $2 \le j \le p$, where p is an integer $3 \le p \le p_1$ and $f_{1s} \equiv 0$, on V_3 for $p + 1 \le s \le n$. Since, φ is a non-vanishing function, there is an open subset V_4

of V_3, where $\varphi_{,j} \neq 0$ for $j = 1, \dots, p$. It follows from (37) that on V_4,

$$\begin{aligned} f_{1j}\varphi_{,k} &= f_{jk}\varphi_{,1}, \quad j \neq k,\ 2 \leq j, k \leq p, \\ f_{12}\varphi_{,s} &= f_{1s}\varphi_{,2}, \quad p+1 \leq s \leq n, \\ f_{kj}\varphi_{,s} &= f_{sj}\varphi_{,k}, \quad j \neq k,\ 2 \leq j, k \leq p,\ p+1 \leq s \leq n, \\ f_{ks}\varphi_{,r} &= f_{sr}\varphi_{,k}, \quad s \neq r,\ p+1 \leq s, r \leq n. \end{aligned}$$

From the first equality we get that $f_{jk} \equiv 0$ on V_4. From the second one we conclude that $\varphi_{,s} \equiv 0$ on V_4. It follows from the third one that $f_{sj} \equiv 0$ and from the last equality we conclude that $f_{sr} \equiv 0$ on V_4. Hence, φ depends on the variables $x_1, \dots, x_p$, and it satisfies the differential equation $(36)_3$ for $1 \leq i \neq j \leq p$,where all f_{ij} do not vanish on V_4. It follows from Lemma A that, on each connected component $W \subset V_4$, where $1 \leq i \neq j \leq \prod_{i \neq j \neq k} f_{ij}\varphi_{,k} \neq 0$, there exist nonconstant differentiable functions $U_i(x_i)$, $1 \leq i \leq p$ such that

$$f_{ij}/p_2 = \varepsilon\, U_i'(x_i)\, U_j'(x_j), \quad 1 \leq i \neq j \leq p_1.$$

where $\varepsilon = 1$ or $\varepsilon = -1$ for all $i \neq j$. We now consider on W the change of variables $y_i = U_i(x_i)$. In this new coordinates $\varphi(y_1, \dots, y_p)$ satisfies PDE's

$$\varphi_{y_i y_j} = \varepsilon\, \varphi, \quad \text{for all } i \neq j.$$

Lemma B implies that φ is given by (13) or (14) on W, according to the value of ε. Moreover, the diagonal elements of the tensor T, $f_{ii}(x_1, \dots, x_p)$ are determined by $(36)_1$. In both cases, one can extend the domain of φ to a subset of V where the functions U_i are defined and φ does not vanish. The converse in both cases is a straightforward computation. $\square$

Proof of Corollary 4. For all cases (i)–(iv) we define $u = \varphi^{1 - p_1 p_2/n}$.

(i) By Theorem 1, the mixed scalar curvature for the metric $\tilde g$ is $\widetilde{K}_{1,2} = -p_1 p_2[\lambda + 2(a_2 - a_1)\, \mu(x)]$. Substituting in (4) with $K = 0$, we get (15).

(ii) It follows from (10)(a) that $\widetilde{K}_{1,2} = p_2\, e^{2U}[U'' - (p_1{-}1)U'^2]$ for the metric $\tilde g$ of Theorem 2. Similarly to (i), we obtain (15).

(iii) It follows from (11)(a) that $\widetilde{K}_{1,2} = (v + w)(p_2 v'' {+} p_1 w'') - p_1 p_2(v'^2 {+} w'^2)$ for the metric $\tilde g$ of Theorem 3. Similarly to (i), we obtain (15).

(iv) Using $\varphi_{,i} = U_i'(af - bf^{-1})$ and $\varphi_{,ii} = (U_i'' + U_i'^2)(af - bf^{-1})$, we get $|\nabla\varphi|^2 = \sum_j U_i'^2 (af - bf^{-1})^2$ and $\Delta^{(1)}\varphi = \sum_j (U_i'' + U_i'^2)(af - bf^{-1})$. From (13) we get the required $\tilde K_{1,2}$ for $\tilde g$ of Theorem 4. Hence u satisfies (15). $\square$

Next we will prove results of Section 2.

Given p_1-dimensional plane D_1 in $\mathbb{R}^n$, denote by $U(D_1)$ the set of all p_1-dimensional planes of $\mathbb{R}^n$ uniquely projecting onto D_1. Taking orthonormal basis e_i $(1 \leq i \leq p_1)$ of D_1, and extending it to orthonormal basis e_i $(1 \leq i \leq n)$ of $\mathbb{R}^n$, we represent any $\tilde D \in U(D_1)$ as a linear graph over D_1 with values in orthogonal complement, i.e., by the system $x_j = \sum_{i=1}^{p_1} a_{ji} x_i$ $(p_1 < j \leq n)$. The

p_1p_2 elements of the matrix $A = (a_{ji})$ can be chosen as local coordinates on real Grassmannian $G_{p_1}(\mathbb{R}^n)$ in a neighborhood $U(D_1)$, in particular, $\dim G_{p_1}(\mathbb{R}^n) = p_1p_2$. To any one-parameter variation $D_1(s)$ of D_1 there corresponds the matrix-function $A(s) = (a_{ji}(s))$. Assuming $A(s)$ of a class C^2 with $A_1 = (dA/ds)(0)$ and $A_2 = (d^2A/ds^2)(0)$, we have $A(s) = sA_1 + \frac{1}{2}s^2A_2 + o(s^2)$. For $\tilde{D} \in U(D_1)$, the stationary values $0 \le \alpha_1 \le \ldots \le \alpha_{p_1} \le \pi/2$ of angle between unit vectors in $\tilde{D}$ and D_1 are called the *angles* between these planes. It is known that there exist orthonormal bases e_i $(1 \le i \le p_1)$ of D_1 and $\tilde{e}_j$ $(1 \le j \le p_1)$ of $\tilde{D}$ such that $\langle e_i, \tilde{e}_j\rangle = \cos\alpha_i\delta_{ij}$ (the property can be taken as the definition of angles α_i). The angles α_i $(1 \le i \le p_1)$ determine the relative position of two p_1-dimensional planes in $\mathbb{R}^n$, and $\rho(\tilde{D}, D_1) = (\sum_{i=1}^{p_1}\alpha_i^2)^{1/2}$ is the *distance*.

Let $p_1 \le p_2$. Choosing the basis, one may represent a variation $D_1(s)$ of D_1 using the matrix $A(s) = sA_1 + \frac{1}{2}s^2A_2 + o(S^2)$, where A_1 consists of $a^{(1)}_{ji} = \delta_{ji}\cos\alpha_i$ $(1 \le i \le p_1,\ p_1 < j \le n)$.

Proof of Theorem 5. Let $\varphi : M \to \mathbb{R}$ be a smooth nonnegative function with a local support $V \subset M$. Assume that $p_1 \le p_2$. Due to discussion above, any variation $D_1(s)$ $(|s| < \varepsilon)$ on V can be represented by orthogonal vector fields $e_i(s) = e_i\cos(s\,\omega_i\,\varphi) + \xi_i\sin(s\,\omega_i\,\varphi)$, where e_i $(i \le p_1)$ is orthonormal basis of $D_{1|\,V}$, ξ_j $(j \le p_2)$ orthonormal basis in $D_2 = D_1^\perp$ on V, and $\omega_i \ge 0$ $(i \le p_1)$ are real numbers. Set $\xi_i(s) = \xi_i\cos(s\,\omega_i\,\varphi) - e_i\sin(s\,\omega_i\,\varphi)$ for $1 \le i \le p_1$ and $\xi_j(s) = \xi_j$ for $p_1 < j \le n$. Indeed, $\xi_j(s)$ $(1 \le j \le p_2)$ span $D_1(s)^\perp$ on V. We have $\frac{de_i}{ds}(0) = \varphi\,\omega_i\,\xi_i$ and $\frac{d\xi_i}{ds}(0) = -\varphi\,\omega_i\,e_i$ on V for $i \le p_1$.

We extend distributions outside of V as $D_1(s) = D_1$ and define $I(s) = I_K(D_1(s))$ for all s. The mixed scalar curvature of $D_1(s)$ over V is

$$K(s) := K(D_1(s), D_1^\perp(s)) = \sum\nolimits_{i=1}^{p_1}\sum\nolimits_{j=1}^{p_2} g(R(e_i(s), \xi_j(s))\xi_j(s), e_i(s)).$$

Derivation by s at $s = 0$ (using symmetries of curvature tensor) yields on V

$$\begin{aligned}\tfrac{d}{ds}K(s)_{|\,s=0} &= 2\,\varphi\sum\nolimits_{i,j=1}^{p_1}[g(R(e_i,\xi_j)\xi_j,\omega_i\xi_i) + g(R(e_i,\xi_j)(-\omega_je_j),e_i)]\\ &= 2\,\varphi\sum\nolimits_{i=1}^{p_1}\omega_i(\mathrm{Ric}_1 - \mathrm{Ric}_2)(e_i,\xi_i).\end{aligned}$$

Notice that $K(s) = K(0)$ outside of V. Hence

$$I'(0) = 2\int_V \varphi\sum\nolimits_{i=1}^{p_1}\omega_i(\mathrm{Ric}_1 - \mathrm{Ric}_2)(e_i,\xi_i)\,d\,\mathrm{vol}\,.$$

Since e_i, ξ_i, ω_i and φ are arbitrary, from above it follows that D is critical for I_K if and only if $(\mathrm{Ric}_1 - \mathrm{Ric}_2)(x, y) = 0$ for all $x \in D$ and $y \perp D$, the first part of the claim has been proved.

Assume now that D_1 is a critical point for I_K. Consider arbitrary variation $D_1(s)$ $(|s| < \varepsilon)$ of D_1 with a local support $V \subset M$. As above, take orthonormal basis e_i $(i \le p_1)$ of D_1 and orthonormal basis ξ_i $(i \le p_2)$ of $D_1^\perp$ on V such that $D_1(s)$ is spanned by $e_i(s)$ and $D_1(s)^\perp$ is spanned by $\xi_j(s)$ as above, where

$\varphi : M \to \mathbb{R}$ a smooth nonnegative function with a support in V. In addition, $\frac{d^2 e_i}{ds^2}(0) = -\varphi^2\,\omega_i^2\, e_i$ and $\frac{d^2\xi_i}{ds^2}(0) = -\varphi^2\,\omega_i^2\,\xi_i$ on V for $i \le p_1$. The second derivative of $K(s)$ at $s = 0$ on V ($K(s) = K(0)$ outside of V) is

$$\begin{aligned}\tfrac12 \tfrac{d^2}{ds^2} K_{\,|\,s=0} &= \varphi^2\big\{ \sum_{j=1}^{p_1} \omega_j^2(\mathrm{Ric}_1 - \mathrm{Ric}_2)(\xi_j, \xi_j) - \sum_{i=1}^{p_1} \omega_i^2(\mathrm{Ric}_1 - \mathrm{Ric}_2)(e_i, e_i) \\ &\quad + 2 \sum_{i,j=1}^{p_1} \omega_i\,\omega_j [\, g(R(e_i, e_j)\xi_i, \xi_j) + g(R(e_i, \xi_j)\xi_i, e_j)\,]\big\}.\end{aligned}$$

Hence $I''(0) = 2\int_V \varphi^2 \sum_{i,j=1}^{p_1} \Phi_{ij}(q)\,\omega_i\,\omega_j\, d\,\mathrm{vol}$, where $\Phi_{ij}(q)$ is defined in (16) and $\omega_i \in \mathbb{R}$ are arbitrary numbers. From above it follows that $I''(0) > 0$ when the form $\mathcal{I}$ is quasi-positive. $\square$

Proof of Proposition 3. Define the linear operators $C_2^i : D_2 \to D_2$ by $C_2(\xi, \cdot) = \sum_i \xi_i C_2^i$ for $\xi = \sum_i \xi_i e_i \in D_1$. Note that $p_2 H_2 = -\sum_i \sigma_1(C_2^i)\, e_i$. Hence $p_2^2 |H_2|^2 = \sum_i \sigma_1^2(C_2^i)$ and $\sigma_1((C_2^i)^2) = \sum_{\alpha,\beta} C^i_{2,\alpha\beta} C^i_{2,\beta\alpha}$.
From $-g(\nabla_{e_\alpha} e_\beta, e_i) = g(e_\beta, \nabla_{e_\alpha} e_i) = C^i_{2,\alpha\beta}$, we get $(\nabla_{e_\alpha} e_\beta)^\top = -\sum_i C^i_{2,\alpha\beta}\, e_i$. Hence

$$\begin{gathered}|B_2|^2 - |T_2|^2 = \textstyle\sum_{\alpha,\beta} g((\nabla_{e_\alpha} e_\beta)^\top, (\nabla_{e_\beta} e_\alpha)^\top) = \\ \textstyle\sum_{\alpha,\beta} g\big(\sum_i C^i_{2,\alpha\beta}\, e_i, \sum_j C^j_{2,\alpha\beta}\, e_j\big) = \sum_i \sum_{\alpha,\beta} C^i_{2,\alpha\beta} C^i_{2,\beta\alpha} = \sum_i \sigma_1((C_2^i)^2).\end{gathered}$$

From known identity $2\,\sigma_2(C_2^i) = \sigma_1^2(C_2^i) - \sigma_1((C_2^i)^2)$, it follows from above that $2\sum_i \sigma_2(C_2^i) = p_2^2|H_2|^2 + |T_2|^2 - |B_2|^2$. Similarly, for $\eta = \sum_\alpha \eta_\alpha e_\alpha$ we get $C_1(\eta, \cdot) = \sum_\alpha \eta_\alpha C_1^\alpha$. Thus, $2\sum_\alpha \sigma_2(C_1^\alpha) = p_1^2|H_1|^2 + |T_1|^2 - |B_1|^2$ and

$$2\sum\nolimits_i \sigma_2(C_2^i) + 2\sum\nolimits_\alpha \sigma_2(C_1^\alpha) = p_1^2|H_1|^2 + |T_1|^2 - |B_1|^2 + p_2^2|H_2|^2 + |T_2|^2 - |B_2|^2. \tag{38}$$

Using Lemma 1.5 of [9], we obtain at $q \in M$

$$\sigma_2(D_1) + \sigma_2(D_2) = \sum\nolimits_i \sigma_2(C_2^i) + \sum\nolimits_\alpha \sigma_2(C_1^\alpha). \tag{39}$$

The integral formula in [11] shows us that

$$\int_M \big[K_{1,2} + |B_1|^2 - p_1^2|H_1|^2 - |T_1|^2 + |B_2|^2 - p_2^2|H_2|^2 - |T_2|^2\big]\, d\,\mathrm{vol} = 0, \tag{40}$$

where H_i, T_i, B_i are the mean curvature vector, the integrability tensor and the 2-nd fundamental form of D_i. The required formula follows from (38)–(40). $\square$

Proof of Proposition 4. We represent the norm of $\nabla\tilde D_1$ using co-nullity operators C_1, C_2

$$|\nabla\tilde D_1|^2 = \sum\nolimits_{i,j,\alpha} (C^\alpha_{1,ij})^2 + \sum\nolimits_{i,\alpha,\beta} (C^i_{2,\alpha\beta})^2. \tag{41}$$

Hence $|\nabla\tilde D_1|^2 = |\nabla\tilde D_2|^2$. Clearly, this expression does not depend on the adapted local orthonormal basis. Let $\tilde C = \{\tilde C_{ij}\}_{1\le i,j\le p}$ be a real matrix of order $p \ge 2$. An elementary calculation shows that

$$\begin{gathered}(p-1)\sum\nolimits_{i,j} (\tilde C_{ij})^2 = \sum\nolimits_{i<j} (\tilde C_{ii} - \tilde C_{jj})^2 + \sum\nolimits_{i<j} (\tilde C_{ij} + \tilde C_{ji})^2 \\ +(p-2)\sum\nolimits_{i\ne j} (\tilde C_{ij})^2 + 2\sum\nolimits_{i<j} (\tilde C_{ii}\,\tilde C_{jj} - \tilde C_{ij}\,\tilde C_{ji}).\end{gathered} \tag{42}$$

Note that the last term in (42) equals $\sigma_2(\tilde{C})$. Hence $\sum_{i,j}(\tilde{C}_{ij})^2 \geq \frac{2}{p-1}\,\sigma_2(\tilde{C})$. Applying this inequality to the matrices C_1^α and C_2^i, we obtain

$$\sum\nolimits_{\alpha;\,i,j}(C^\alpha_{1,ij})^2 \geq \tfrac{2}{p_1-1}\sum\nolimits_\alpha \sigma_2(C^\alpha_1), \quad \sum\nolimits_{i;\,\alpha\,\beta}(C^i_{2,\alpha\,\beta})^2 \geq \tfrac{2}{p_2-1}\sum\nolimits_i \sigma_2(C^i_2),$$

and, in view of (41), $|\nabla\tilde{D}_1|^2 \geq \frac{2}{p_1-1}\sum_\alpha \sigma_2(C^\alpha_1) + \frac{2}{p_2-1}\sum_i \sigma_2(C^i_2)$. From that the inequality (19) follows. □

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